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One Block Two Point Parameter Dependent Integration Formula for Stiff Initial Value Problems

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ABSTRACT

In this paper, a 2-point Block Parameter Dependent Integration Formula (BPDIF) is developed. The method is dependent on the parameter τ that is inserted to help determine the range of values for which the method is A-stable. Range of values for τ is presented herein. The proposed method are of order p=2 and compete favorably with other known methods in literature.

1.0 INTRODUCTION

Numerical integration of Stiff Initial Value Problems (IVPs) in Ordinary Differential Equation (ODEs) of the form:

$$y' = f(x,y), \quad y(a) = y_0, a \le x \le b, y: R \to R^N \quad f: R \times R^N \to R^N \quad (1)$$

is still an active area of research. Most times, the solution to Eq. (1) cannot be obtained analytically, so methods like the Linear Multistep formula (LMF) are developed to integrate Eq. (1). Using LMF requires that the method must have k-1 initial values for integration to commence and the method must be A-stable. Over the years, many numerical analysts have taken various steps in solving Eq. (1), some of the existing methods are documented in [5-9].

To exploit computational speedup inherent in modern day computers, the Block Methods were introduced [4]. Block Methods are generalization of Linear Multistep Formulas and generate approximate solutions at more than one grid point at every cycle of integration. The number of points, depends on the structure of the block method. A – stability been a very important requirement for method designed for the integration of stiff initial value problems.In order to ensure A-stability of the method, the integration formula being developed must be implicit. In this paper, a family of two point block method is introduced that is parameterized by the insertion of parameter. The insertion of parameter is to afford for search of range of interval for which the family of method is A – stable.

2.0 STRUCTURE OF THE PAPER

The rest of the paper is structured as follows: section two is on the derivation of the proposed method, the stability characteristics are presented in the third section while in section four is on the numerical experiment on two standard numerical test problems. In section five, the conclusion is presented.

2.1 Derivation of Method

Consider the method of the form:

$$A_0 Y_{n+1} = A_1 Y_n + hB(F_{n+1} + \tau F_n) \quad (2)$$

where τ is a real value constant determined in such a way as to ensure that Eq. (2) is zero stable. If $\tau = 0$, (2) reduces to a 2-Point Block Backward Differentiation Formula (BDF) developed in [9]. Also, A_0, A_1 and B are 2×2 matrices whose entries are as prescribed as follows:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad and \quad B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}.$$

In Eq. (2), Y_{n+1}, Y_n, F_n and F_{n+1} vectors are given by:

$$Y_{n+1} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, \quad Y_n = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, \quad F_n = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} and \; F_{n+1} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$

Note that $y_n \approx y(x_n)$ and $f_n \approx f(x_n, y_n)$.

The proposed method Eq. (2) is herein referred to as two-point Block Parameter Dependent Integration formula (BPDIF).

The Linear difference operator $L[y(x);h]$ associated with Eq. (2) is

[redacted] (3)

Inserting coefficient matrices and vectors in Eq. (3)

yields Eq. (4)

$$L[y(x);h]=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}y(x_{n+1})\\y(x_{n+2})\end{bmatrix}-\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}y(x_{n-1})\\y(x_n)\end{bmatrix}-\begin{bmatrix}b_{11}&0\\0&b_{22}\end{bmatrix}\left(\begin{bmatrix}f(x_{n+1},y(x_{n+1}))\\f(x_{n+2},y(x_{n+2}))\end{bmatrix}+\tau\begin{bmatrix}f(x_{n-1},y(x_{n-1}))\\f(x_n,y(x_n))\end{bmatrix}\right)$$

$$=\begin{pmatrix}y(x_{n+1})-a_{11}y(x_{n-1})-a_{12}y(x_n)-hb_{11}(f(x_{n+1},y(x_{n+1}))+\tau f(x_{n-1},y(x_{n-1})))\\y(x_{n+2})-a_{21}y(x_{n-1})-a_{22}y(x_n)-hb_{22}(f(x_{n+2},y(x_{n+2}))+\tau f(x_n,y(x_n)))\end{pmatrix}$$

(4)

It is assumed here that $y(x_n)$ is differentiable and continuous for as many times as maybe required.

Taylor expanding $y(x_{n+1}),\ y(x_{n-1}),\ y(x_n),\ f(x_{n+1},y(x_{n+1})),\ f(x_{n-1},y(x_{n-1}))$ in first row of $L[y(x);h]$ in Eq. (4) about x_n, gives

$$L[y(x);h]=y(x_n)+hy'(x_n)+h^2\frac{y''(x_n)}{2!}+h^3\frac{y'''(x_n)}{3!}+\ldots+h^q\frac{y^q(x_n)}{q!}-a_{11}y(x_n)+a_{11}hy'(x_n)-$$
$$a_{11}h^2\frac{y''(x_n)}{2!}+a_{11}h^3\frac{y'''(x_n)}{3!}+\ldots+a_{11}(-1)^q h^q\frac{y^q(x_n)}{q!}-a_{12}y(x_n)-b_{11}h(y'(x_n)+hy''(x_n)+$$
$$h^2\frac{y'''(x_n)}{2!}+\ldots+h^q\frac{y^{q+1}(x_n)}{q!}+\alpha y'(x_n)-\alpha hy''(x_n)+\alpha h^2\frac{y'''(x_n)}{2!}+\ldots+(-1)^q h^q\frac{y^{q+1}(x_n)}{q!}$$

(5)

Simplifying (5) to obtain

$$L_1[y(x);h]=(1-a_{11}+a_{21})y(x_n)+h(1+a_{11}-b_{11}-\tau b_{11})y'(x_n)+h^2\left(\frac{1}{2!}-\frac{a_{11}}{2!}-b_{11}+\tau b_{11}\right)y''(x_n)+$$
$$h^3\left(\left(\frac{1}{3!}-\frac{a_{11}}{3!}-\frac{b_{11}}{2!}+\tau\frac{b_{11}}{2!}\right)y'''(x_n)+\ldots+h^p\left(\frac{1}{p!}-\frac{a_{11}}{p!}-\frac{b_{11}}{(p-1)}+\tau\frac{b_{11}}{(p-1)}\right)y^p(x_n)\right)$$

(6)

$$=C_{10}y(x_n)+hC_{11}y'(x_n)+h^2C_{12}y''(x_n)+\ldots+h^pC_{1p}y^p(x_n)+\ldots$$

(7)

Where

$$C_{10}=1-a_{11}-a_{21}$$

$$C_{11}=1+a_{11}-b_{11}-\tau b_{11}$$

$$C_{12}=1-a_{11}-2b_{11}+2\tau b_{11}$$

$$C_{1p}=\frac{1}{p!}+(-1)^p\frac{a_{11}}{p!}-\frac{b_{11}}{(p-1)!}+(-1)^p\tau\frac{b_{11}}{(p-1)!}; \qquad (8)$$

Similarly, Taylor expanding

$$y(x_{n+1}),\ y(x_{n-1}),\ y(x_n),\ f(x_{n+1},y(x_{n+1})),\ f(x_{n-1},y(x_{n-1}))$$

(9)

in second row of $L[y(x);h]$ in Eq. (4) about x_n, gives

Simplifying Eq. (9) to get

$$L_2[y(x);h]=(1-a_{11}+a_{21})y(x_n)+h(2+a_{11}-b_{11}-\tau b_{11})y'(x_n)+$$
$$h^2\left(\frac{4}{2!}-\frac{a_{11}}{2!}+2b_{11}\right)y''(x_n)+h^p\left(\frac{2^p}{p!}-\frac{a_{11}}{p!}-\frac{2^pb_{11}}{(p-1)}\right)y^p(x_n)+\ldots$$

(10)

$$L_2[y(x);h]=C_{20}y(x_n)+hC_{21}y'(x_n)+h^2C_{22}y''(x_n)+\ldots+h^pC_{2p}y^p(x_n)+\ldots$$

(11)

Where

$$C_{20}=1-a_{11}-a_{21}$$
$$C_{21}=2+a_{11}-b_{11}-\tau b_{11}$$
$$C_{22}=4-a_{11}-4b_{11} \qquad \ldots$$

$$C_{2p}=\frac{2^p}{p!}+(-1)^p\frac{a_{11}}{p!}-\frac{2^{p-1}b_{11}}{(p-1)!} \qquad (12)$$

Equations Eqs. (8) and (12) give the order condition for the method Eq. (2).

Definition 1 cf. [3]

A block method is said to be of order p if

$$c_0=c_1=c_2=\ldots=c_p=0,\ c_{p+1}\neq 0 \qquad (13)$$

Where c_{p+1} is the error constant.

To determine the order of the BPDIF, substitute a_{11}, a_{12} and b_{11} into Eq. (8)

$$C_{10}=\frac{-3+\tau-1+3\tau+4-4\tau}{-3+\tau}=0$$

$$C_{11}=\frac{-3+\tau+1-3\tau+2+2\tau}{-3+\tau}=0$$

$$C_{12} = \frac{-3+\tau-1+3\tau+4-4\tau}{-3+\tau} = 0$$

$$C_{13} = \frac{-3+\tau+1-3\tau+6+6\tau}{-3+\tau} \neq 0$$

Since $C_{10} = C_{11} = C_{12} = 0, C_{13} \neq 0$, the equation is of order p=2.

Similarly, substitute a_{21}, a_{22} and b_{22} into Eq. (12)

$$C_{20} = \frac{5+\tau+4-4\tau-9+3\tau}{5+\tau} = 0$$

$$C_{21} = \frac{10+2\tau-4+4\tau-6+6\tau}{5+\tau} = 0$$

$$C_{21} = \frac{10+2\tau-4+4\tau-6+6\tau}{5+\tau} = 0$$

$$C_{22} = \frac{10+2\tau+2-2\tau-12}{5+\tau} = 0$$

$$C_{23} = \frac{40+8\tau-4+4\tau-72}{3!(5+\tau)} \neq 0$$

Since $C_{20} = C_{21} = C_{22} = 0, C_{23} \neq 0$, the equation is of order p=2.

The linear systems of equations Eqs. (8) and (12) are solved to obtain the elements of matrices A and B.

$$y_{n+1} + \frac{1-3\tau}{-3+\tau} y_{n-1} - \frac{4(-1+\tau)}{-3+\tau} y_n = -\frac{2}{-3+\tau} h(f_{n+1} + \tau f_{n-1})$$
(14)

$$y_{n+2} - \frac{4(-1+\tau)}{5+\tau} y_{n-1} + \frac{3(-3+\tau)}{5+\tau} y_n = \frac{6}{5+\tau} h(f_{n+2} + \tau f_n) \tag{15}$$

Coupling equations Eqs. (14) and (15) in the form of (2), observe

$$A_1 = \begin{bmatrix} \frac{1-3\tau}{-3+\tau} & \frac{4(-1+\tau)}{-3+\tau} \\ \frac{4(-1+\tau)}{5+\tau} & -\frac{3(-3+\tau)}{5+\tau} \end{bmatrix} \tag{16}$$

$$B = \begin{bmatrix} -\frac{2}{-3+\tau} & 0 \\ 0 & \frac{6}{5+\tau} \end{bmatrix} \tag{17}$$

2.2 Stability of the Method

The BPDIF is said to be zero stable if all the roots of the first characteristics polynomial associated with Eq. (2), $\rho(\zeta)$ has root $|\zeta| \leq 1, \quad t = 1,2.$ and root $|\zeta| = 1$ is simple, [4].

$$= -\frac{7}{(-3+\tau)(5+\tau)} + \frac{22\zeta}{(-3+\tau)(5+\tau)} - \frac{15\zeta^2}{(-3+\tau)(5+\tau)} + \frac{2\tau}{(-3+\tau)(5+\tau)} - \frac{4\zeta\tau}{(-3+\tau)(5+\tau)} + \frac{2\zeta^2\tau}{(-3+\tau)(5+\tau)} - \frac{7\tau^2}{(-3+\tau)(5+\tau)} + \frac{6\zeta\tau^2}{(-3+\tau)(5+\tau)} + \frac{\zeta^2\tau^2}{(-3+\tau)(5+\tau)} \tag{18}$$

The characteristic polynomial $\rho(\zeta)$ associated with Eq. (2) is

$$\rho(\zeta) = |A_0\zeta - A_1|$$

$$\zeta_1 \to 1, \zeta_2 \to \frac{-7+2\tau-\tau^2}{-15+2\tau+\tau^2}.$$

To ensure that method (2) is zero stable, determine a range of values for parameter τ for which

$$\left| \frac{-7+2\tau-\tau^2}{-15+2\tau+\tau^2} \right| < 1.$$

The choice of τ in Eq. (18) that ensures that method Eq. (2) is zero stable exist in interval $\tau \in (-1,1)$.

Applying BPDIF Eq. (2) to the test equation

$$y' = \lambda y,$$

yields the characteristic polynomial

$$\pi(R,z) = \det(RA_0 - A_1 - zB(R+\tau)) \tag{19}$$

The boundary locus for some values of τ in the region $(-1,1)$ is shown in Figure 1. Figure1(a) shows the stability region for the Integration formulae Eq. (2) when $\tau = (0, 0.9)$ while Figure 1(b) displays the stability region for the values of the parameter $\tau = (-0.9, 0)$

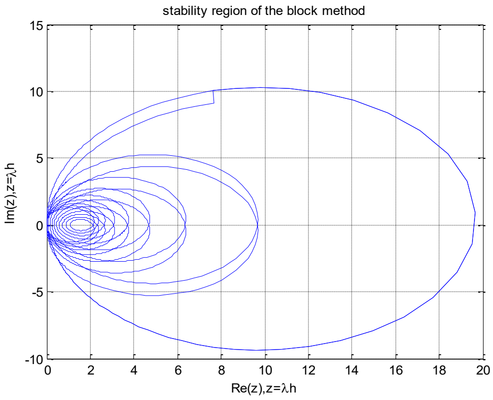

Figure 1(a): stability region of BPDIF when $\tau = 0(0)0.9$

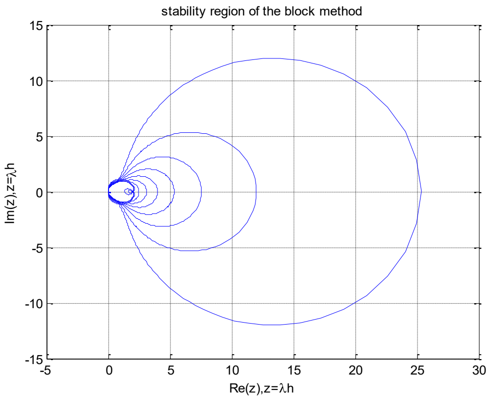

Figure 1(b): stability region of BPDIF when $\tau = -0.9(-0.9)0$

The region of absolute stability in the Figure1contains the entire left plane of the complex plane. Hence, BPDIF is A-Stable for the range of τ chosen.

3.0 NUMERICAL EXPERIMENTS

In this section, the BPDIF is used to solve the following numerical problems:

Problem 1 cf. [1]

Given the nonlinear system

$$y_1' = -2y_1 + y_2 + 2Sinx$$

$$y'_2 = 998y_1 - 999y_2 + 999(Cosx - Sinx)$$

with initial values

$$y_1(0) = 2, \quad y_2(0) = 3$$

The analytical solutions for the system are:

$$y_1(x) = 2e^{-x} + Sinx, \quad y_2(x) = 2e^{-x} + Cosx.$$

Problem 2 cf. [4]

The system of linear equations represented below:

$$y' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} y \text{ where } y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The analytic solution is

$$y(x) = \begin{pmatrix} e^{-0.1x} \\ e^{-10x} \\ e^{-100x} \\ e^{-1000x} \end{pmatrix} \quad for\ 0 \le x \le 1$$

The approximate solutions of problems 1-2 are generated by the BPDIF method. The exact solutions of the problems and the absolute error consequential to the use of the BPDIF method are presented in the following tables.

Table1: The absolute error result for problem 1 using BPDIF

x	y_i	Exact Solution	Absolute Error
0.25	y_1	1.805005525	0.001364753381670
	y_2	2.526513988	0.003798467595267
0.5	y_1	1.692486858	0.000929169705516
	y_2	2.090643881	0.003643782237286
1.0	y_1	1.577229867	0.000599187936153
	y_2	1.276061188	0.003362281843368
2.0	y_1	1.179967993	0.001286263318653
	y_2	−0.145476270	0.002270576004271
4.0	y_1	-0.720171217	0.00147057026510
	y_2	-0.617012343	-0.001350166288186
6.0	y_1	-0.274457993	-0.001589081746467
	y_2	0.9651277911	-0.000225713244751
8.0	y_1	0.9900291719	0.000017984049149
	y_2	−0.1448291085	0.001703997625869
10.0	y_1	−0.5439303110	0.001602160734157
	y_2	−0.8389807292	-0.001164466199138

Table 1 highlights the error difference of the exact solution of problem to the approximate solution generated by BPDIF Eq. (2) for $\tau = -0.1$.

Table 2: Absolute error for Problem 2

x	Exact Solution	Absolute Error
0.25	0.082084998623899	0.002637659259219
0.5	0.006737946999085	0.000298063557151
1.0	0.000045399929762	0.000003086784812

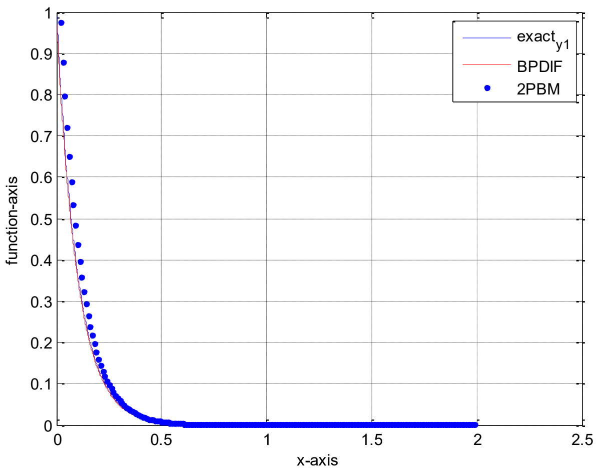

Figure 2: Solution curve to problem 2 using BPDIF and 2PBM

The figure 2 shows the graphical behavior of the proposed method BPDIF in comparison with the 2PBM developed in [2] for problem 1. The BPDIF approximates the closest to the exact solution of the problem.

4.0 CONCLUSION

In this paper, a family of 2-point parameter dependent block integration formulae has been developed. The proposed family of method is A-stable for $\tau \in (-1,1)$ and performs creditably when compared to existing method in the literature.

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